Boundary Representations on C^*- algebras

with Matrix Units

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AMS (1970) Classifications. Primary 46 L 05, 46 L 15;
Secondary 46 J 10

Key words and phrases. C*-algebra, boundary representation, matrix units, Choquet boundary.

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Abstract

Let $\mathfrak{A}$ be a C*-algebra with unit, let $\mathfrak{S}$ be a linear subspace of $\mathfrak{A} \otimes M_n$ which contains the natural set of matrix units and which generates $\mathfrak{A}$ as a C*-algebra. Let $\mathfrak{T}$ be the subset of $\mathfrak{A}$ consisting of entries of matrices in $\mathfrak{S}$. Then the boundary representations of $\mathfrak{A} \otimes M_n$ relative to $\mathfrak{S}$ are parametrized by the boundary representations of $\mathfrak{A}$ relative to $\mathfrak{T}$. Also, a non-trivial example is given of a subalgebra of a C*-algebra which possesses exactly one boundary representation.

The concept of Choquet boundary has recently been generalized by Arveson [1,2] to apply to an arbitrary C*-algebra with unit and a linear subspace (or even subset) which generates the C*-algebra. More specifically, he defines the notion of an (irreducible) boundary representation of the C*-algebra relative to the generating subspace. (The precise definition is given below.) In the event that the algebra is abelian, and so of the form $C(X)$, we may identify the points of X with the irreducible representations of $C(X)$; a point in X will be a boundary representation for a subspace $\mathfrak{S}$ of $C(X)$ if and only if it lies in the Choquet boundary for $\mathfrak{S}$ (see [1], p.168). Since the boundary representations for a subspace reveal information on the extent to which the subspace determines the structure of the C*-algebra (cf. [1], Theorem 2.2.5), it becomes useful to be able to find the boundary representations for a given subspace.

In this paper we shall study the problem of finding boundary representations on C*-algebras which possess a set of $n \times n$ matrix units, i.e. C*-algebras of the form $\mathfrak{A} \otimes M_n$. Provided that we assume that a linear subspace $\mathfrak{S}$ "contains the constants" in the sense that it contains the set of matrix units, we can determine the boundary representations of $\mathfrak{A} \otimes M_n$ relative to $\mathfrak{S}$; they are described in terms of the boundary representations on $\mathfrak{A}$ relative to an appropriately chosen subspace of $\mathfrak{A}$. Furthermore, we give a non-trivial example of a subalgebra of a C*-algebra which possesses exactly one boundary representation. This indicates a possible scarcity of boundary representations.

We now give all the necessary definitions and some background material. If A is a C*-algebra with unit, e, then $A \otimes M_n$ is the C*-algebra consisting of all $n \times n$ matrices with entries in A. If A acts on a Hilbert space H then $A \otimes M_n$ is taken to act on the Hilbert space $H \oplus \ldots \oplus H$ (n factors) in the usual way. If φ is a linear map from one C*-algebra, A, into another, B, then we let $\varphi^{(n)}$ denote the map from $A \otimes M_n$ into $B \otimes M_n$ defined by $\varphi^{(n)}(a_{ij}) = (\varphi(a_{ij}))$, where (a_{ij}) is a matrix in $A \otimes M_n$. The map φ is said to be <u>completely positive</u> if each $\varphi^{(n)}$ is positive. A fundamental theorem of Stinespring [4] says that if $\varphi: A \longrightarrow B(H)$ is completely positive then there exists a representation π of A acting on a Hilbert space K and a bounded linear map $V: H \to K$ such that $\varphi(a) = V^*\pi(a)V$ for all $a \in A$. Further, π and V may be chosen so that the range of V is cyclic for π, i.e. so that $K = [\pi(A)VH]$. Note also that if $\varphi(e) = I$ then V is an isometry.

Arveson's definition of boundary representation is the following: let A be a C*-algebra with unit and let S be a subset of A which contains the unit and which generates A as a C*-algebra. (We write $A = C^*(S)$.) Then an irreducible representation π of A, acting on the Hilbert space H, is a <u>boundary representation</u> for S if π is the only completely positive linear map of A into $B(H)$ which extends the restriction $\pi|S: S \to B(H)$.

Note that if φ and ψ are two completely positive linear maps on A which agree on a subset S then they agree on the smallest norm closed and self-adjoint linear subspace R which contains S. (This follows from the fact that a positive linear map preserves adjoints and is automatically continuous.) As a consequence, $\mathfrak{S}$ and $\mathfrak{R}$ have precisely the same set of boundary representations. This triviality allows for various reformulations of many results about boundary representations; for example, it is often possible to replace assumptions on a given subset $\mathfrak{S}$ with the same or similar assumptions on the norm closed, self-adjoint, linear subspace generated by $\mathfrak{S}$. In this paper we shall not bother to take advantage of this possibility and further, we shall generally take $\mathfrak{S}$ to be a linear subspace in its own right.

We establish some further notation. If $\mathfrak{B}$ is a C*-algebra with unit I, we say that a family of operators $\{F_{ij}\}$ $i,j = 1,2,\ldots,n$ is a set of $n \times n$ <u>matrix units</u> provided:

(1) $F_{ij} = F_{ji}^*$, all i,j.

(2) $F_{ij}F_{kl} = \delta_{jk}F_{il}$, all i,j,k,l

(where $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ otherwise)

(3) $\Sigma_{i=1}^{n} F_{ii} = I$.

It is routine to show that if $\mathfrak{B}$ possesses a set of $n \times n$ matrix units $\{F_{ij}\}$ then $\mathfrak{B}$ can be written in the form (more precisely: $\mathfrak{B}$ is *-isomorphic to) $\mathfrak{A} \otimes M_n$, where for $\mathfrak{A}$ we may take the C*-algebra $F_{11}\mathfrak{B}F_{11}$ with unit F_{11}. Consequently, we shall generally deal with algebras of the form $\mathfrak{A} \otimes M_n$. We also define a set of n^2 linear maps $E_{ij}: \mathfrak{A} \to \mathfrak{A} \otimes M_n$ $(i,j = 1,2,\ldots,n)$ as follows: if $a \in \mathfrak{A}$, let $E_{ij}(a)$ be that matrix in $\mathfrak{A} \otimes M_n$ whose i,j-entry is a and whose other entries are all 0.

Finally, we recall the definition of the <u>compression</u> of an operator C on a Hilbert space $\mathfrak{H}$ to a closed linear subspace $\mathfrak{M}$ of $\mathfrak{H}$; it is merely the restriction to $\mathfrak{M}$ of the operator PCP , where P is the orthogonal projection on $\mathfrak{M}$. By the compression of a representation we mean the simultaneous compression to a fixed subspace of all the operators in the image of the representation.

Use of Stinespring's theorem allows one to reduce problems about boundary representations to problems about compressions. For example, if we wish to prove that π is a boundary representation of $\mathfrak{A}$ for a subspace $\mathfrak{S}$, we let φ be a completely positive extension to $\mathfrak{A}$ of the restriction $\pi|\mathfrak{S}$. It is necessary to show that $\varphi = \pi$; by Stinespring's theorem $\varphi = V^*\sigma V$ for some representation σ and isometry V. So φ is unitarily equivalent to the compression of σ to the range of V. This compression is known for elements of $\mathfrak{S}$; it is necessary to extend this knowledge to $\mathfrak{A}$. Before applying this technique we state a fairly simple lemma about compressions and matrix units.

__Lemma.__ Let $\mathfrak{A}$ be a C*-algebra, let $\{F_{ij}\}$ be a set of $n \times n$ matrix units in $\mathfrak{A}$, and let π be a representation of $\mathfrak{A}$ acting on a Hilbert space $\mathfrak{H}$. Let $\mathfrak{M}$ be a closed subspace of $\mathfrak{H}$ with P the orthogonal projection on $\mathfrak{M}$. If $\{P\pi(F_{ij})P\}$ is a set of matrix units (in the C*-algebra $P\pi(\mathfrak{A})P$, with unit P) then $\mathfrak{M}$ is invariant under each $\pi(F_{ij})$.

__Proof.__ Let $G_{ij} = P\pi(F_{ij})P$. Each G_{kk} is a sub-projection of P, and if α is a vector in the range of G_{kk} then $\alpha = G_{kk}\alpha = P\pi(F_{kk})\alpha$. Hence it follows that

$$\|\alpha\|^2 \geq \|\pi(F_{kk})\alpha\|^2 = \|P\pi(F_{kk})\alpha\|^2 + \|(I-P)\pi(F_{kk})\alpha\|^2$$
$$= \|\alpha\|^2 + \|(I-P)\pi(F_{kk})\alpha\|^2$$

As a consequence, $(I-P)\pi(F_{kk})\alpha = 0$ and $\pi(F_{kk})$ acts as the identity on the range of G_{kk}. Further, since $I = \Sigma_i \pi(F_{ii})$, we have

$$G_{kk} = \Sigma_i \pi(F_{ii})G_{kk} = G_{kk} + \Sigma_{i \neq k}\pi(F_{ii})G_{kk},$$

and so $\Sigma_{i\neq k}\pi(F_{ii})G_{kk} = 0$. If we multiply on the left by $\pi(F_{jj})$ we obtain $\pi(F_{jj})G_{kk} = 0$ for $j \neq k$. This holds for each k, so we have shown that each G_{kk} is invariant under each $\pi(F_{jj})$ and hence $P = \Sigma_k G_{kk}$ is invariant under each $\pi(F_{jj})$.

We now show that P is invariant under each $\pi(F_{ij})$, $i \neq j$. Fix i and j and let $A = F_{ij} + F_{ji} + \sum_{k\neq i,j} F_{kk}$. Then A is unitary and so is $\pi(A)$. We claim that $\pi(A)$ leaves $\mathfrak{M}$ invariant. If α is a vector in the range of G_{kk} for some $k \neq i,j$ then $P\pi(A)\alpha = (G_{ij} + G_{ji} + \sum_{k\neq i,j} G_{kk})\alpha = \alpha$ and, as in the paragraph above, it follows that $(I-P)\pi(A)\alpha = 0$. So $\pi(A)$ leaves G_{kk} invariant for $k \neq i,j$. If α is in the range of G_{ii} then $P\pi(A)\alpha = G_{ji}\alpha$, so $\|\alpha\| = \|G_{ji}\alpha\| = \|P\pi(A)\alpha\| \leq \|\pi(A)\alpha\| = \|\alpha\|$ and we obtain $\pi(A)\alpha = G_{ji}\alpha$. In particular, $\pi(A)\alpha \in \mathfrak{M}$. The same holds for α in the range of G_{jj} and so, since $P = \Sigma G_{kk}$, we obtain the invariance of $\mathfrak{M}$ under $\pi(A)$.

In exactly the same fashion we can prove that if $B = F_{ij} - F_{ji} + \sum_{k\neq i,j} F_{kk}$ then $\pi(B)$ leaves $\mathfrak{M}$ invariant. Making use of the invariance of $\mathfrak{M}$ under each $\pi(F_{kk})$ we find that $\pi(F_{ij}) + \pi(F_{ji})$ and $\pi(F_{ij}) - \pi(F_{ji})$ both leave $\mathfrak{M}$ invariant.

Finally, it follows that $\pi(F_{ij})$ leaves $\mathfrak{M}$ invariant and the lemma is proven.

We remark in passing that it does not follow from the hypotheses of the lemma that $\mathfrak{M}$ is invariant under π.

Let $\mathfrak{A}$ be a C*-algebra with identity e and let $\mathfrak{A}^{(n)} = \mathfrak{A} \otimes M_n$. It is well known that if ρ is a representation of $\mathfrak{A}^{(n)}$ then there is a representation π of $\mathfrak{A}$ such that ρ is unitarily equivalent to $\pi^{(n)}$. (An indication of the proof: take for the space of π the range of the projection $\rho(E_{11}(e))$; let $\pi(a)$ be the restriction to this space of the operator $\rho(E_{11}(a))$, for all $a \in \mathfrak{A}$.) This establishes a one-to-one correspondence between the unitary equivalence classes of representations of $\mathfrak{A}^{(n)}$ and the equivalence classes of representations of $\mathfrak{A}$. Since we are only interested in representations up to unitary equivalence, we shall always take representations of $A^{(n)}$ to be of the form $\pi^{(n)}$. Note also that $\pi^{(n)}$ is irreducible if and only if π is irreducible. The following theorem shows that, with respect to suitable linear subspaces, the property of being a boundary representation is also preserved by the correspondence $\pi \longleftrightarrow \pi^{(n)}$.

<u>Theorem</u>. Let $\mathfrak{A}$ be a C*-algebra with unit e. Let $\mathfrak{A}^{(n)} = \mathfrak{A} \otimes M_n$ and let $\mathfrak{S}$ be a linear subspace of $\mathfrak{A}^{(n)}$ which generates $\mathfrak{A}^{(n)}$ and which contains the set of matrix units $E_{ij}(e)$, $i,j = 1,\ldots,n$. Let $\mathfrak{T}$ be the set of operators in $\mathfrak{A}$ which appear as a matrix entry in some element of $\mathfrak{S}$. Then an irreducible representation π of $\mathfrak{A}$ is a boundary representation for $\mathfrak{T}$ if and only if $\pi^{(n)}$ is a boundary representation for $\mathfrak{S}$.

<u>Proof.</u> We first show the trivial implication that if $\pi^{(n)}$ is a boundary representation for $\mathfrak{S}$ then π is a boundary representation for $\mathfrak{T}$. Suppose π is not a boundary representation for $\mathfrak{T}$. Then let φ be a completely positive extension of $\pi|\mathfrak{T}$ to $\mathfrak{A}$ such that $\varphi \neq \pi$. It follows easily that $\varphi^{(n)}$ is a completely positive extension of $\pi^{(n)}|\mathfrak{S}$ to $\mathfrak{A}^{(n)}$ such that $\varphi^{(n)} \neq \pi^{(n)}$ and hence that $\pi^{(n)}$ is not a boundary representation for $\mathfrak{S}$.

Now assume that π is a boundary representation for $\mathfrak{T}$, acting on the Hilbert space $\mathfrak{H}$. Let φ be a completely positive extension to $\mathfrak{A}^{(n)}$ of the restriction $\pi^{(n)}|\mathfrak{S}$. In order to prove that $\pi^{(n)}$ is a boundary representation for $\mathfrak{S}$ we must show that $\varphi = \pi^{(n)}$. Let $\mathfrak{J} = \mathfrak{H} \oplus \ldots \oplus \mathfrak{H}$ (n factors) denote the space of $\pi^{(n)}$. By Stinespring's theorem there exists a representation σ of $\mathfrak{A}^{(n)}$ acting on a Hilbert space $\mathfrak{K}$ and an isometry $V: \mathfrak{J} \to \mathfrak{K}$ such that $\varphi(A) = V^*\sigma(A)V$ for all $A \in \mathfrak{A}^{(n)}$. For convenience, write $F_{ij} = E_{ij}(e)$, $i,j = 1,\ldots,n$.

Let $P = VV^*$ (the range projection for V) and let $\mathfrak{M}$ be the range of P. We first claim that $\mathfrak{M}$ is invariant under the operators $\sigma(F_{ij})$, $i,j = 1,\ldots,n$. From the lemma, it suffices to show that $\{P\sigma(F_{ij})P\}$ is a set of matrix units in the algebra $P\sigma(\mathfrak{A}^{(n)})P$. But since $F_{ij} \in \mathfrak{S}$ we have $P\sigma(F_{ij})P = VV^*\sigma(F_{ij})VV^* = V\pi^{(n)}(F_{ij})V^*$. V is unitary from $\mathfrak{J}$ onto $\mathfrak{M}$ so $\{P\sigma(F_{ij})P\}$ is a set of matrix units as required. Since $\{\sigma(F_{ij})\}$ is closed under the taking of adjoints, the $\sigma(F_{ij})$ all commute with P.

We now let $t \in \mathfrak{T}$ and compute $\varphi(E_{kl}(t))$ for arbitrary $k,l = 1,\ldots,n$. Let T be an element of $\mathfrak{S}$ such that t is an entry in the matrix T, say the i,j-entry. It is trivial to see that $E_{kl}(t) = F_{ki}TF_{jl}$, whence $\sigma(E_{kl}(t)) = \sigma(F_{ki})\sigma(T)\sigma(F_{jl})$. Then, using the fact that each $\sigma(F_{ij})$ commutes with P, we have $P\sigma(E_{kl}(t))P = P\sigma(F_{ki})P\sigma(T)P\sigma(F_{jl})P$. And finally, since $\varphi = V^*\sigma V$ agrees with $\pi^{(n)}$ on $\mathfrak{S}$, we obtain

$$\begin{aligned}
\varphi(E_{kl}(t)) &= V^*\sigma(E_{kl}(t))V \\
&= V^*P\sigma(E_{kl}(t))PV \\
&= V^*\sigma(F_{ki})VV^*\sigma(T)VV^*\sigma(F_{jl})V \\
&= \pi^{(n)}(F_{ki})\pi^{(n)}(T)\pi^{(n)}(F_{jl}) \\
&= \pi^{(n)}(F_{ki}TF_{jl}) \\
&= \pi^{(n)}(E_{kl}(t)).
\end{aligned}$$

Any matrix in $\mathfrak{A}^{(n)}$ with entries from $\mathfrak{T}$ is a sum of matrices of the form $E_{kl}(t)$, $t \in \mathfrak{T}$. Therefore, we have shown that φ must agree with $\pi^{(n)}$ on the subset of all matrices in $\mathfrak{A}^{(n)}$ with entries from $\mathfrak{T}$ and we have not, as yet, used the assumption that π is a boundary representation of $\mathfrak{A}$ for $\mathfrak{T}$.

Consider the mapping $\psi\colon \mathfrak{A} \to \mathfrak{B}(\mathfrak{H})$ given by

$$\psi(a) = \pi^{(n)}(F_{11})\varphi(E_{11}(a))\pi^{(n)}(F_{11})|\text{range } \pi^{(n)}(F_{11}), \text{ for } a \in A.$$

Since we identify $\mathfrak{H}$ and the range of $\pi^{(n)}(F_{11})$, $\psi(a)$ is an operator on $\mathfrak{H}$. ψ is obviously linear and it is completely positive since it is the composition of completely positive maps. (Namely, ψ is the composition of E_{11}, φ, and the mapping which takes an operator on $\mathfrak{J}$ to its compression to the subspace $\mathfrak{H}$ considered as the first summand in $\mathfrak{J}$. E_{11} is completely positive since it is a $*$-homomorphism; φ is assumed to be completely positive; and compression mappings are always completely positive.) Further, it is clear from the results above that $\psi|\mathfrak{T} = \pi|\mathfrak{T}$. Therefore, by the assumption that π is a boundary representation for $\mathfrak{T}$, we have $\psi = \pi$.

If $a \in \mathfrak{A}$ then $\psi(a)$ is just the 1,1-entry in the matrix $\varphi(E_{11}(a))$. We now know that this entry is $\pi(a)$; to show that φ acts as $\pi^{(n)}$ does on $E_{11}(a)$ we must show that all the other entries of $\varphi(E_{11}(a))$ are 0.

Now, $\varphi(E_{11}(a))$ is an operator on the Hilbert space $\mathfrak{J} = \mathfrak{H} \oplus \ldots \oplus \mathfrak{H}$. Let $\tau: \mathfrak{B}(\mathfrak{J}) \rightarrow \mathfrak{B}(\mathfrak{H})$ be defined by $\tau(A_{ij}) = \Sigma_{i=1}^{n} A_{ii}$, where (A_{ij}) is an element of $\mathfrak{B}(\mathfrak{J})$ written as a matrix with entries in $\mathfrak{B}(\mathfrak{H})$ in the natural way. Each mapping of the form $(A_{ij}) \rightarrow A_{kk}$ is a compression (to $\mathfrak{H}$ considered as the k^{th} summand of $\mathfrak{J}$) and hence is completely positive. τ is a sum of such maps and so is also completely positive. Hence the map $\tau \circ \varphi \circ E_{11}$ of $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$ is completely positive and is easily seen to extend $\pi|\mathfrak{T}$. Again, since π is a boundary representation, $\tau \circ \varphi \circ E_{11} = \pi$. Thus if $a \in \mathfrak{A}$, and if we let (A_{ij}) be the matrix form of $\varphi(E_{11}(a))$, then $\pi(a) = \tau(\varphi(E_{11}(a))) = A_{11} + \Sigma_{i=2}^{n} A_{ii}$. But we showed above that $A_{11} = \pi(a)$, hence $\Sigma_{i=2}^{n} A_{ii} = 0$. Suppose further, for the moment, that $a \geq 0$ in $\mathfrak{A}$. Then $\varphi(E_{11}(a)) = (A_{ij}) \geq 0$ and hence each $A_{ii} \geq 0$. Since $\Sigma_{i=2}^{n} A_{ii} = 0$ it follows that $A_{ii} = 0$ for $i = 2,\ldots,n$. Using this we can show further that $A_{ij} = 0$ for all pairs $i \neq j$. Indeed, since $(A_{ij}) \geq 0$, it is the square of some self-adjoint element (B_{ij}) of $\mathfrak{B}(\mathfrak{J})$. Then for $i \geq 2$ we have

$$0 = A_{ii} = \Sigma_{j=1}^{n} B_{ij}B_{ji} = \Sigma_{j=1}^{n} B_{ij}B_{ij}^{*},$$

from which it follows that $B_{ij} = 0$ for all $i \geq 2$. If $j \geq 2$ then $B_{ij} = B_{ji}^{*} = 0$ also, so B_{11} is the only non-zero entry of (B_{ij}) and consequently A_{11} is the only non-zero entry of (A_{ij}). Thus we have proven that if $a \in Ⓐ$ and $a \geq 0$ then $\varphi(E_{11}(a)) = \pi^{(n)}(E_{11}(a))$. But any element of Ⓐ is a linear combination of positive elements, so $\varphi(E_{11}(a)) = \pi^{(n)}(E_{11}(a))$ for all $a \in Ⓐ$.

It is now a simple matter to show that $\varphi(E_{ij}(a)) = \pi^{(n)}(E_{ij}(a))$ for all $i,j = 1,\ldots,n$ and all $a \in Ⓐ$. Since $E_{ij}(a) = F_{i1}E_{11}(a)F_{1j}$ and since Ⓜ is invariant under all the $\sigma(F_{ij})$, we have

$$\begin{aligned}
&\varphi(E_{ij}(a)) = V^{*}\sigma(E_{ij}(a))V \\
&= V^{*}\sigma(F_{i1})\sigma(E_{11}(a))\sigma(F_{1j})V \\
&= V^{*}\sigma(F_{i1})VV^{*}\sigma(E_{11}(a))VV^{*}\sigma(F_{1j})V \\
&= \pi^{(n)}(F_{i1})\pi^{(n)}(E_{11}(a))\pi^{(n)}(F_{1j}) \\
&= \pi^{(n)}(E_{ij}(a)).
\end{aligned}$$

And finally, since an arbitrary element in $Ⓐ^{(n)}$ is just a sum of elements of the form $E_{ij}(a)$ with $a \in Ⓐ$, we have proven that $\varphi = \pi^{(n)}$ on $Ⓐ^{(n)}$. Thus $\pi^{(n)}$ is a boundary representation for Ⓢ and the theorem is proven.

Attention is drawn to the special case in which Ⓐ is abelian and hence may be taken to be $C(X)$, the algebra of continuous complex valued functions on some compact Hausdorff space X. $C(X) \otimes M_n$ can be interpreted either as $n \times n$ matrices with entries in $C(X)$ or as the algebra of continuous functions from X into M_n. The irreducible representations of both $C(X)$ and $C(X) \otimes M_n$ are just the point evaluations and so can be identified with X in either case. For each $x \in X$ then, let π_x be point evaluation on $C(X)$ and ρ_x be point evaluation on $C(X) \otimes M_n$. (In the previous notation, $\rho_x = \pi_x^{(n)}$.) If Ⓣ is a linear subspace of $C(X)$ containing the constants then $\partial_{Ⓣ}(X) = \{x \in X \mid \pi_x$ is a boundary representation for Ⓣ$\}$ is just the usual Choquet boundary for Ⓣ. If Ⓢ is a linear subspace of $C(X) \otimes M_n$ containing the constants (i.e. the constant matrices), if Ⓣ is the linear subspace of $C(X)$ generated by the entries of the matrices in Ⓢ and if $\partial_{Ⓢ}(X) = \{x \in X \mid \rho_x$ is a boundary representation for Ⓢ$\}$, then the theorem above states simply that $\partial_{Ⓢ}(X) = \partial_{Ⓣ}(X)$.

The following example shows that the condition that the subspace Ⓢ of Ⓐ $\otimes M_n$ contain the matrix units cannot be dropped. Take $X = [0,1]$, the unit interval, and Ⓐ $= C(X)$. Let $f: [0,1] \to \mathbb{R}$ be a strictly positive, strictly increasing, continuous function. (For example, $f(t) = t+1$ will do.) We consider Ⓐ $\otimes M_2$ and let Ⓢ be the linear subspace generated by the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the matrix $F = \begin{bmatrix} 0 & 0 \\ f & 0 \end{bmatrix}$. (As it happens, Ⓢ is even a sub-algebra of Ⓐ $\otimes M_2$.) For Ⓣ we take the subspace of $C(X)$ generated by the constant function 1 and the function f. From the Stone-Weierstrass theorem it follows that C^*(Ⓣ) is all of $C(X)$. We sketch the argument which shows that C^*(Ⓢ) is all of $C(X) \otimes M_2$. Since f is real, $\begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}$ is the adjoint of F and so is in C^*(Ⓢ). The products in either order of F and F^*, namely the matrices $\begin{bmatrix} f^2 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & f^2 \end{bmatrix}$, are also in C^*(Ⓢ). Now the function f^2 separates points and never vanishes; it follows that polynomials in f^2 <u>without constant term</u> are dense in $C(X)$. Hence, by taking norm limits of polynomials without constant terms of the matrices $\begin{bmatrix} f^2 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & f^2 \end{bmatrix}$ we obtain the fact that all matrices of the form $\begin{bmatrix} g & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & g \end{bmatrix}$ are in $C^*(\mathbb{S})$, where g is an arbitrary continuous function on X. Appropriate multiplications of such matrices by F or F^* yield matrices with arbitrary functions in the off diagonal positions and it follows immediately that $C^*(\mathbb{S}) = \mathbb{A} \otimes M_2$.

For $t \in X$ we let π_t and ρ_t be the corresponding irreducible representations of $C(X)$ and $C(X) \otimes M_2$ respectively. The boundary representations of $C(X)$ relative to $\mathbb{T}$ are just π_0 and π_1. (This is easy to show directly, or cf. [3], section 8.) A glance at the first paragraph of the theorem shows that no use was made of the matrix units in that paragraph, so we can conclude that if $t \neq 0,1$ then ρ_t is not a boundary representation of $C(X) \otimes M_2$ for $\mathbb{S}$. We now show that ρ_0 is not a boundary representation while ρ_1 is.

There are several ways of constructing a completely positive extension of $\rho_0|\mathbb{S}$ which is unequal to ρ_0. One of them is the following: let $a \in [0,1]$ with $a \neq 0$. Let $\sigma = \rho_a \oplus \rho_a$. Since ρ_a acts on $\mathbb{C}^2$, σ acts on $\mathbb{C}^4$. We choose two orthogonal unit vectors in $\mathbb{C}^4$: let $v_1 = (f(0)/f(a),0,0,(1-(f(0)/f(a))^2)^{\frac{1}{2}})$ and $v_2 = (0,1,0,0)$. Let P be the orthogonal projection on the span of v_1 and v_2. For each $G \in \mathbb{A} \otimes M_2$ let $\psi(G)$ be the compression of $\sigma(G)$ to the range of P. We identify the range of P with $\mathbb{C}^2$ by taking $\{v_1,v_2\}$ as the standard basis; operators in the range of ψ are then 2×2 matrices expressed with respect to this basis. It is obvious that ψ is completely positive; we need merely show that it extends $\rho_0|Ⓢ$ but is unequal to ρ_0.

Purely routine calculations show that $\sigma(F)v_1 = f(0)v_2$ and $\sigma(F)v_2 = 0$. Hence $\psi(F)$ = the matrix of the compression of $\sigma(F)$ to range $P = \begin{bmatrix} 0 & 0 \\ f(0) & 0 \end{bmatrix} = \rho_0(F)$. It then follows that ψ extends ρ_0. It is also easy to see that ψ and ρ_0 take different values at the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, so $\psi \neq \rho_0$. Thus ρ_0 is not a boundary representation.

It remains now to show that ρ_1 is a boundary representation for Ⓢ. Let φ be a completely positive extension of ρ_1. Use Stinespring's theorem to write $\varphi = V^*\pi V$, where π is a representation of Ⓐ $\otimes M_2$ acting on some Hilbert space Ⓗ, V is an isometric linear mapping of $\underset{\sim}{C}^2$, the space of ρ_1, into Ⓗ, and the range of V is cyclic for π. Let $e_1 = (1,0)$ and $e_2 = (0,1)$ in $\underset{\sim}{C}^2$ and let $v_1 = Ve_1$ and $v_2 = Ve_2$. Finally, let $P = VV^*$ = the range projection of V. Our given information is that, with respect to the basis $\{v_1, v_2\}$, the matrix of the compression of $\pi(F)$ to P is just $\begin{bmatrix} 0 & 0 \\ f(1) & 0 \end{bmatrix}$.

Let $G = F + F^* = \begin{bmatrix} 0 & f \\ f & 0 \end{bmatrix}$. Then since $\varphi(G) = \varphi(F) + \varphi(F)^* = \begin{bmatrix} 0 & f(1) \\ f(1) & 0 \end{bmatrix}$ and since $\varphi = V^*\pi V$, we know that the compression of $\pi(G)$ to P has the matrix $\begin{bmatrix} 0 & f(1) \\ f(1) & 0 \end{bmatrix}$. This means that $\pi(G)v_1 = f(1)v_2 + \omega$, where ω is some vector orthogonal to P. But $\|\pi(G)\| \leq \|G\| = f(1)$, hence it follows that $\omega = 0$. So $\pi(G)v_1 = f(1)v_2$ and in exactly the same way $\pi(G)v_2 = f(1)v_1$. In particular, $\pi(G)$ leaves P invariant. Replace G by $H = F - F^*$ and repeat the argument to show that $\pi(H)$ also leaves P invariant. But then both $\pi(F)$ and $\pi(F^*)$ leave P invariant. Since Ⓐ $\otimes M_2$ is generated by F and the identity, P is invariant under the representation π. But the range of V was assumed to be cyclic for π, hence it is all of $\mathfrak{H}$. Thus V is actually a unitary operator and so φ is not merely completely positive, it is a representation of $\mathfrak{A} \otimes M_2$. Since it agrees with ρ_1 on a generating set, it is equal to ρ_1. Thus ρ_1 is the only boundary representation of $\mathfrak{A} \otimes M_2$ for the sub-algebra S.

Perhaps the main significance of this example is that it indicates a possible scarcity (or lack?) of boundary representations. (Cf. [2], section 2.1.) In any event it seems to rule out an existence proof for boundary representations along lines analogous to that for the Choquet boundary, that is by identifying the boundary representations as the extreme points of a suitable compact convex set and invoking the Krein-Milman theorem.

References

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